On-Line Algorithms and Reverse Mathematics

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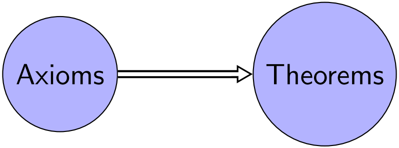
Reverse Mathematics

Given a theorem, what axioms are necessary to prove the theorem?

Most mathematical activity:



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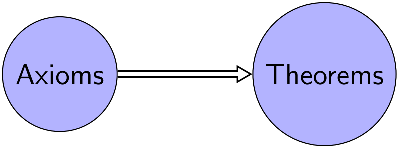

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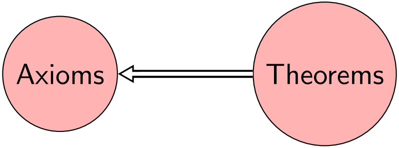

Second-Order Arithmetic

We work in *second order arithmetic.*

Two types of variables:

- Number variables x, y, z, w ranging over elements of $\mathbb{N}$
- Set variables X, Y, Z, W ranging over subsets of $\mathbb{N}$

From this, can encode:

- Real numbers
- Functions $f : \mathbb{N} \to \mathbb{N}$
- Borel sets
- Graphs, other combinatorial objects

Reverse Mathematics: Important Subsystems

$\mathsf{ACA}_0 \to \mathsf{WKL}_0 \to \mathsf{RCA}_0$

RCA_0: Recursive Comprehension Axiom.

- Comprehension for Δ^0_0 (computable) sets plus Σ^0_1-induction.
- Mild assumptions, used as base theory

WKL_0: Weak König's Lemma. Every infinite binary tree has an infinite path.

Over RCA_0, equivalent to:

- Heine-Borel Theorem
- Continuous functions on a closed interval are integrable
- Every locally k-colorable graph is k-colorable

Reverse Mathematics: Important Subsystems

ACA_0: Arithmetical Comprehension Axiom

Over RCA_0, equivalent to:

- Every countable vector space has a basis
- Ramsey's theorem for sets of size ≥ 3
- Existence of a range of arbitrary $f : \mathbb{N} \to \mathbb{N}$
- Existence of Halting set relative to A:

$$A' = \{e : \Phi_e^A(e)\downarrow\}$$

Reverse Mathematics: Important Subsystems

ACA_0: Arithmetical Comprehension Axiom

Over RCA_0, equivalent to:

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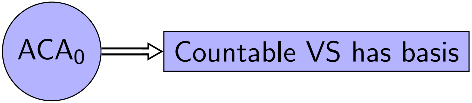

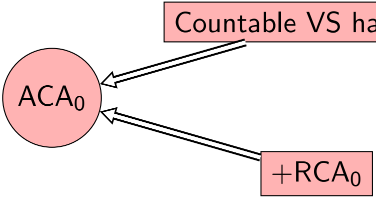

$$
\begin{tikzcd}
\Pi^0_1\text{--}\mathsf{CA}_0 \arrow[d, Rightarrow] & & \\
\mathsf{ATR}_0 \arrow[d, Rightarrow] & & \\
\mathsf{ACA}_0 \arrow[ddd, Rightarrow] \arrow[dddr, Rightarrow] \arrow[drr, Rightarrow] & & \\
 & & \mathsf{I}\Sigma^0_n \arrow[d, Rightarrow] \\
 & & \mathsf{B}\Sigma^0_n \arrow[d, Rightarrow] \\
\mathsf{WKL}_0 \arrow[dd, Rightarrow] & \mathsf{RT}^2_2 \arrow[ddddl, Rightarrow] \arrow[ddddr, Rightarrow] & \cdots \arrow[d, Rightarrow] \\
 & & \mathsf{I}\Sigma^0_3 \arrow[d, Rightarrow] \\
\mathsf{WWKL}_0 \arrow[dd, Rightarrow] & & \mathsf{B}\Sigma^0_3 \arrow[d, Rightarrow] \\
 & & \mathsf{I}\Sigma^0_2 \arrow[d, Rightarrow] \\
\mathsf{DNR} \arrow[d, Rightarrow] & & \mathsf{B}\Sigma^0_2 \arrow[dll, Rightarrow] \\
\mathsf{RCA}_0 & &
\end{tikzcd}
$$

Sequential Problems

Problem: Every finite graph without odd cycles is bipartite.

Sequential Problem: For every infinite sequence of finite graphs without odd cycles, there exists an infinite sequence of bipartitions.

Pigeonhole Principle: Given $k \geq 2$ and $f : A \to k$, with $|A| < \infty$, there is y such that $|\{x \in A : f(x) = y\}| \geq \dfrac{|A|}{k}$.

Sequential Pigeonhole Principle: Given $k \geq 2$ and a sequence $\langle A_n, f_n \rangle_{n \in \mathbb{N}}$, $f_n : A_n \to k$ and $|A_n| < \infty$, there is a sequence $\langle y_n \rangle_{n \in \mathbb{N}}$ such that

$$\forall n \left(|\{x \in A_n : f_n(x) = y_n\}| \geq \frac{|A_n|}{k} \right).$$

Sequential Problems

Question: Which sequential problems are equivalent to ACA_0, WKL_0, or RCA_0 over RCA_0?

Proposition

The following are equivalent over RCA_0*:*

(i) ACA_0

(ii) *Given a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of finite sets, there is a sequence of upper bounds $\langle b_n \rangle_{n \in \mathbb{N}}$ such that $\forall n \forall x\, (x \in X_n \rightarrow x \leq b_n)$.*

Proof: Can compute range of $f : \mathbb{N} \to \mathbb{N}$

$X_n = \{0, s+1\}$ if $f(s) = n$; $\quad X_n = \{0\}$ otherwise.

Motivating Example: Graph Colorings

Let $\mathcal{C}$ be a universal class of graphs (closed under isomorphisms and finite subgraphs) such that every finite graph in $\mathcal{C}$ is r-colorable.

Sequential Problem: For every sequence of finite graphs $\langle G_n \rangle_{n \in \mathbb{N}}$, $G_n = (V_n, E_n) \in \mathcal{C}$, there exists a sequence of proper r-colorings $\langle \chi_n \rangle_{n \in \mathbb{N}}$, $\chi_n : V_n \to r$.

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Strength of this problem?

Theorem (Gasarch and Hirst)

$\mathsf{WKL}_0 \leftrightarrow$ *Every locally r-colorable graph is r-colorable.*

View graph coloring as a game

Alice plays vertex v_i, chooses connections with $v_0, \ldots, v_{i-1}$.

- If resulting graph not in $\mathcal{C}$, Alice loses.

Bob plays color c_i.

- If resulting coloring is improper, Bob loses.

Alice	v_0		v_1		v_2		$\cdots$
Bob		c_0		c_1		c_2	$\cdots$

Definition

The class $\mathcal{C}$ of graphs is *on-line r-colorable* if Bob has a winning strategy in this game.

The Class of Forests is NOT On-line 2-Colorable

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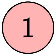

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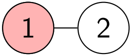

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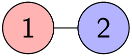

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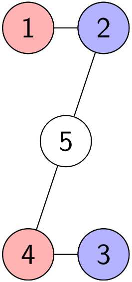

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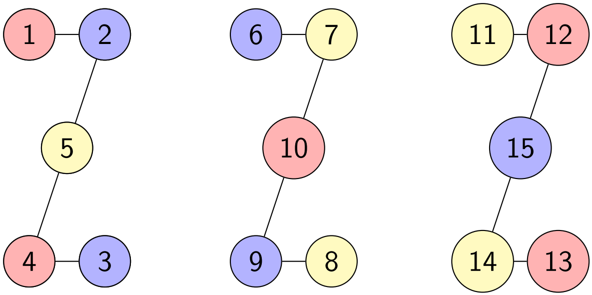

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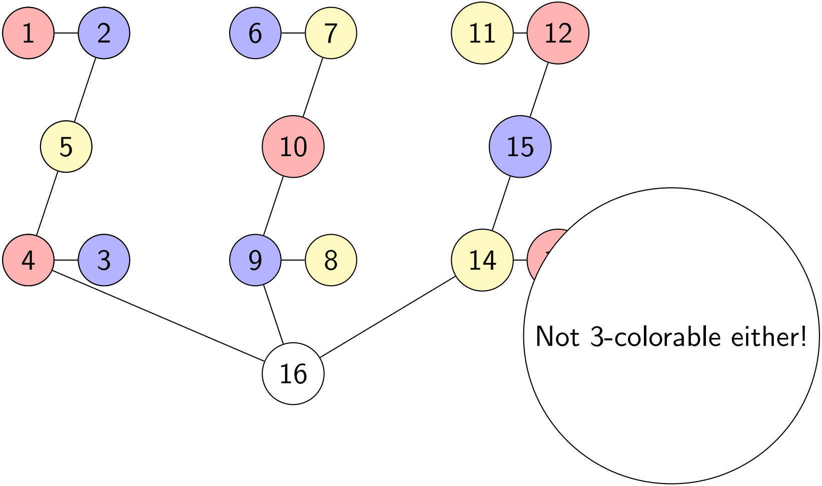

Sequential Problems

A *problem* is a triple (A, B, R) where A and B are trees and $R \subseteq A \otimes B$.

$\mathrm{P}(A, B, R)$ is the statement

$$\forall X(\alpha(X) \to \exists Y \beta(X, Y))$$

where:

- $\alpha(X)$ holds if X is a finite set of the form $\{(0, s_0, a_0), \ldots, (k-1, s_{k-1}, a_{k-1})\}$ where $s_0 < \cdots < s_{k-1}$ and $\langle a_0, \ldots, a_{k-1}\rangle \in A$, and
- $\beta(X, Y)$ holds if Y is a finite set of the form $\{(0, t_0, b_0), \ldots, (k-1, t_{k-1}, b_{k-1})\}$ where $t_0 < \cdots < t_{k-1}$ and $\langle a_0, \ldots, a_{k-1}\rangle \; R \; \langle b_0, \ldots, b_{k-1}\rangle$ holds.

Sequential Problems

$\mathsf{P}(A, B, R)$ is the statement

$$\forall X(\alpha(X) \to \exists Y \beta(X, Y))$$

$\mathsf{SeqP}(A, B, R)$ is the statement

$$\forall X\, (\forall n\, \alpha(X_n) \;\to\; \exists Y\, \forall n\, \beta(X_n, Y_n))$$

Here $X = \langle X_n \rangle_{n \in \mathbb{N}}$, $Y = \langle Y_n \rangle_{n \in \mathbb{N}}$.

View problems as games

$\bar{a} = \langle a_0, \ldots, a_{k-1} \rangle \in A$ Questions by Alice

$\bar{b} = \langle b_0, \ldots, b_{k-1} \rangle \in B$ Responses by Bob

Game $\mathrm{G}(A, B, R)$ is played as follows: Alice and Bob alternate:

$$\begin{array}{r|cccccccc} \text{Alice} & a_0 & & a_1 & & a_2 & & \cdots \\ \hline \text{Bob} & & b_0 & & b_1 & & b_2 & \cdots \end{array}$$

Alice can stop the game at any time.

Bob is required to respond to every one of Alice's plays.

If k rounds, Bob wins if either $\langle a_0, \ldots, a_{k-1} \rangle \notin A$ or $\langle a_0, \ldots, a_{k-1} \rangle \; R \; \langle b_0, \ldots, b_{k-1} \rangle$ holds; otherwise Alice wins.

View problems as games

Game $\mathsf{G}(A, B, R)$ is played as follows: Alice and Bob alternate:

$$\begin{array}{r|ccccccc} \text{Alice} & a_0 & & a_1 & & a_2 & & \cdots \\ \hline \text{Bob} & & b_0 & & b_1 & & b_2 & \cdots \end{array}$$

(A, B, R) is *solvable* if for every $\bar{a} \in A$ there is a $\bar{b} \in B$ such that $\bar{a}\ R\ \bar{b}$ holds.

(A, B, R) is *on-line solvable* if Bob has a winning strategy in $\mathsf{G}(A, B, R)$.

(A, B, R) is *on-line k-solvable* if Bob has a winning strategy in the restricted game $\mathsf{G}_k(A, B, R)$ where Alice is required to stop after the k^{th} round (or earlier).

On-Line Algorithms

On-Line Algorithms have been useful in studying:

- Graph colorings
- Matching/Marriage problems
- Task scheduling problems
- Paging/server problems
- Competitive auctions

Useful whenever we must make a sequence of choices for a series of inputs *immediately as they arrive*, with no future knowledge.

On-line Task Scheduling

Process tasks of time (1, 1, 1, 3, 3, 3, 6) on 3 processors.

Optimal Solution:

3		
1		
1	3	
1	3	6

On-Line Solution:

6		
3	3	3
1	1	1

Competitive ratio $= 10/6 \approx 1.667$.

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Competitive ratio $= 10/6 \approx 1.667$.

Graham's on-line algorithm: Move to the processor with lightest load. Competitive ratio $2 - \frac{1}{k}$

Albers: Best competitive ratio for an online algorithm is in $(1.852, 1.923]$

Bounded problems

(A, B, R) is *semi-bounded* if Bob's valid responses are bounded by a function of Alice's previous plays. More precisely, there is a function f such that if $\langle a_0, \ldots, a_{k-1} \rangle \; R \; \langle b_0, \ldots, b_{k-1} \rangle$ holds, then

$$b_0 < 2^{f\langle a_0 \rangle},\ b_1 < 2^{f\langle a_0, a_1 \rangle},\ \ldots,\ b_{k-1} < 2^{f\langle a_0, a_1, \ldots, a_{k-1} \rangle}$$

(A, B, R) is *bounded* if, in addition to being semi-bounded, there is a function g such that Alice is required to play $a_i < 2^{g(i)}$ for all i.

Proposition (RCA_0)

Let $k \geq 1$ and let (A, B, R) be a bounded problem. Then $\mathsf{G}_k(A, B, R)$ is determined.

Separating RCA_0 from WKL_0

Proposition (RCA_0)

Let (A, B, R) be a problem which is on-line solvable. Then $\mathrm{SeqP}(A, B, R)$ *holds.*

Proof: Use Bob's winning strategy as a uniformly computable procedure.

Theorem (H.) (RCA_0)

Let $k < \omega$, and let (A, B, R) be a bounded problem which is not on-line k-solvable. Then $\mathrm{SeqP}_k(A, B, R)$ *implies* WKL_0.

Diagonal Nonrecursion

DNR: There is a Diagonally Non-Recursive function; For every oracle A, there is $g : \mathbb{N} \to \mathbb{N}$ such that $\forall e \, (g(e) \neq \Phi_e^A(e))$.

DNR(r): There is such a g with range $\{0, \ldots, r-1\}$.

Theorem

For any $r \geq 2$,

$$\mathsf{DNR}(r) \leftrightarrow \mathsf{WKL}_0 \to \mathsf{WWKL}_0 \to \mathsf{DNR} \to \mathsf{RCA}_0.$$

The principle $\mathrm{Predict}_k(r)$

For every oracle A there is a sequence $\langle \Delta_0^A, \ldots, \Delta_{k-1}^A \rangle$ of partial $\Sigma_1^{0,A}$-functions

$$\Delta_i^A : U_{i+1} \to r$$

whose domains are a nested sequence of $\Sigma_1^{0,A}$-sets

$$U_0 = \mathbb{N} \supseteq U_1 \supseteq \ldots \supseteq U_k$$

such that if $\langle f_0, \ldots f_{k-1} \rangle$ is *any* sequence of partial $\Sigma_1^{0,A}$-functions

$$f_i : U_i \to r$$

then there is an $x \in U_k$ such that $(\forall i < k)\ f_i(x) = \Delta_i^A(x)$.

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then there is an $x \in U_k$ such that $(\forall i < k)\ f_i(x) = \Delta_i^A(x)$.

FALSE in the real world. Note that $\mathrm{Predict}_1(r) \leftrightarrow \neg \mathrm{DNR}(r)$.

The principle $\mathsf{Predict}_k(r)$

Theorem (Schmerl, RCA_0)

- *For* $r \geq 2$, $1 \leq k < \omega$, *we have* $\mathsf{Predict}_k(r) \leftrightarrow \neg\mathsf{WKL}_0$.

The principle $\mathsf{Predict}_k(r)$

Theorem (Schmerl, RCA_0)

- *For* $r \geq 2$, $1 \leq k < \omega$, *we have* $\mathsf{Predict}_k(r) \leftrightarrow \neg\mathsf{WKL}_0$.
- $\neg\mathsf{WKL}_0$ *implies the existence of an infinite non-2-colorable forest with finite components.*

  (essentially a sequence of finite forests without a corresponding sequence of 2-colorings).

The principle $\mathrm{Predict}_k(r)$

Theorem (Schmerl, RCA_0)

- *For* $r \geq 2$, $1 \leq k < \omega$, *we have* $\mathrm{Predict}_k(r) \leftrightarrow \neg\mathrm{WKL}_0$.
- $\neg\mathrm{WKL}_0$ *implies the existence of an infinite non-2-colorable forest with finite components.*

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Theorem (Dorais, Hirst, Shafer)

$\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \exists r \mathrm{DNR}(r) \nvdash \mathrm{WKL}_0$.

Conjecture

$\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2 + \exists k(\neg\mathrm{Predict}_k(2)) \nvdash \exists r \mathrm{DNR}(r)$.

Separating RCA_0 from WKL_0

Let $M_k(A, B, R)$ be one more than the largest possible play from either Alice or Bob in the game $G_k(A, B, R)$.
It exists whenever (A, B, R) is bounded.

Theorem (H.) (RCA_0)

Let $k \in \mathbb{N}$. Let (A, B, R) be a bounded problem which is not on-line k-solvable. Let $M = M_k(A, B, R)$. If $\mathsf{Predict}_k(M + 1)$ holds, then $\mathsf{SeqP}_k(A, B, R)$ fails.

Proof in the case of r-coloring a forest

Theorem (Schmerl) (RCA_0)

Let $\mathrm{Col}(\mathcal{F}, r)$ *be the problem of* r*-coloring a finite forest. Let* $k \in \mathbb{N}$ *be such that* $\mathrm{Col}(\mathcal{F}, r)$ *is not on-line* k*-solvable. If* $\mathrm{Predict}_k(r+1)$ *holds, then* $\mathrm{SeqCol}_k(\mathcal{F}, r)$ *fails.*

Proof.

Construct the graph $G_n = (V_n, E_n)$: Put $0 \in V_n$.

If $\Delta_i^A(n)$ converges in exactly s steps, then put $s+1 \in V_n$, and call it v_{i+1}.

By assumption, $\Delta_0^A(n), \ldots, \Delta_i^A(n)$ are defined.
If they list a valid r-coloring of $v_0, \ldots, v_i$, then connect v_{i+1} according to Alice's winning strategy.
If they do not list a valid r-coloring, then do not connect v_{i+1} to any other vertices.

Proof.

Suppose $\langle \chi_n \rangle_{n \in \mathbb{N}}$ is a valid sequence of r-colorings of $\langle G_n \rangle_{n \in \mathbb{N}}$. Define $\langle f_i \rangle_{i<k}$ by $f_i(n) = \chi_n(v_i)$ if v_i exists; $f_i(n) \uparrow$ otherwise.

By $\mathrm{Predict}_k(r+1)$, there is some n with $\Delta_i^A(n) = f_i(n) = \chi_n(v_i)$ for all $i < k$. So χ_n is a valid coloring for all $i < k$, which Bob can use as a winning play contradicting that Alice has a winning strategy. $\square$

Separating WKL_0 from ACA_0

Definition

A problem (A, B, R) has a *solvable closed kernel* if Bob has a winning play such that every initial segment of that play is also winning.

(Technically, the closed kernel R' is a modification of the relation R).

Example: If a graph coloring problem is solvable, its closed kernel is solvable.

Example: The task scheduling problem we saw was solvable, but its closed kernel was not.

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Separating WKL_0 from ACA_0

Example: The Pigeonhole Principle does not have a solvable closed kernel. Consider:

$$(0, 1, 1, 2, 2, 0, 0, 0)$$

Example: Given a sequence in $3^{<\infty}$, find a value that appears over 1% of the time.

$$(0, 1, \ldots, 1, 2, \ldots, 2, 0, \ldots, 0)$$

(10 x) *(100 x)* *(1000 x)*

The closed kernel is not solvable. 0 and 2 are both solutions, but both fail at different initial segments.

Proposition (ACA_0)

Let (A, B, R) *be a solvable problem. Then* $\mathrm{SeqP}(A, B, R)$ *holds.*

Theorem (H., WKL_0)

Let (A, B, R) *be a semi-bounded problem. If the closed kernel* (A, B, R') *is solvable, then* $\mathrm{SeqP}(A, B, R)$ *holds.*

Proof: Dovetail the sequence of requests $\langle n, s, a\rangle$ to get a tree. Solvable closed kernels will ensure that the tree is infinite.

(The tree has height at least $\langle n, 0, 0\rangle$: if $n' < n$, then part of the request $X_{n'}$ will be enumerated by that node, and at least one partial solution will extend to a full solution.)

An infinite branch of the tree will encode a sequence of solutions.

Separating WKL_0 from ACA_0

Theorem (Dorais, H., RCA_0)

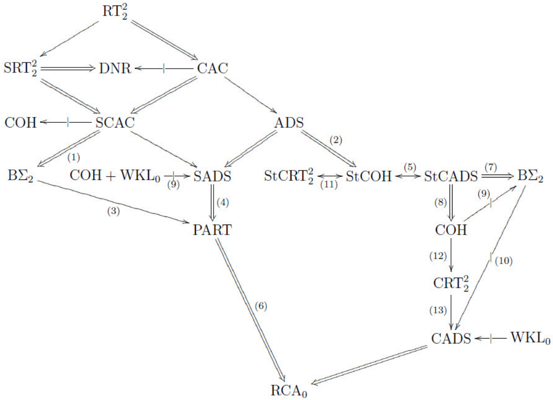
(1) *Let $k < \omega$. If (A, B, R) has a closed kernel that is not k-solvable and $\mathrm{SeqP}_k(A, B, R)$ holds, then ACA_0 holds.*

(2) *If $\mathrm{I}\Sigma^0_2$ holds, then (1) holds for nonstandard $k \in \mathbb{N}$.*

Uses concept called "Good-For-Uniform k-Tuple"

Generalizes Schmerl's "Good Tuple."

Thank you! - The Reverse Math Zoo



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Appendix: Good Tuples

Our proof is based on Schmerl's concept of "Good Tuples."

Let $n \geq 2$. The n-tuple $\langle X_0, X_1, \ldots, X_{n-1} \rangle$ is *good* if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{n-1}$, each X_i is enumerable, and whenever $Y_1, Y_2, \ldots, Y_{n-2}$ are disjoint enumerable sets such that
$Y_i \subseteq X_i$, $1 \leq i \leq n$, and
$X_{i-1} \setminus (Y_1 \cup \cdots \cup Y_i)$ is enumerable, $1 \leq i \leq n$, then
$X_{n-1} \setminus (Y_1 \cup \cdots Y_n) \neq \varnothing$.

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$X_{n-1} \setminus (Y_1 \cup \cdots Y_n) \neq \varnothing$.

DO NOT EXIST in models of ACA_0.
(Take $Y_1 = X_1,\ Y_2 = \cdots = Y_{n-2} = \varnothing$)

Appendix: Good Tuples

Lemma (Schmerl)

Let $(\mathcal{N}, \mathbb{N})$ *be a model,* $n < \omega$. *Then* $\mathcal{N} \vdash \mathsf{ACA}_0$ *holds if and only if* $\mathcal{N} \vdash$ *there are no uniformly good* n*-tuples.*

Lemma (H., RCA_0)

(1) *Let* $n < \omega$. *Then* ACA_0 *holds if and only if there are no uniformly good* n*-tuples.*

(2) *If* $\mathsf{I}\Sigma^0_2$ *holds, then (1) holds for nonstandard* $n \in \mathbb{N}$.

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Theorem (Dorais, H., RCA_0)

Let $k \in \mathbb{N}$, and let (A, B, R) be a problem. If the closed kernel (A, B, R') is not k-solvable and there is a good k-tuple, then $\mathrm{SeqP}_k(A, B, R)$ *fails.*

Proof.

Let $\langle a_0, \ldots, a_{k-1} \rangle$ be a request from Alice such that for any winning response $\langle b_0, \ldots, b_{k-1} \rangle$ (meaning that $\bar{a}\ R\ \bar{b}$), there exists $j < k$ such that $\langle a_0, \ldots, a_{j-1} \rangle\ R\ \langle b_0, \ldots, b_{j-1} \rangle$ fails.
Let $\langle X_0, \ldots, X_{k-1} \rangle$ be a good k-tuple with $X_0 = \mathbb{N}$.
Assume that $\mathrm{SeqP}_k(A, B, R)$ holds.

Define $\langle A_n \rangle_{n \in \mathbb{N}}$ as follows: $(s_i, a_i) \in A_n$ if and only if $e_{X_i}(s_i) = n$.
So the sequence of requests in A_n will be $\langle a_0, \ldots, a_i \rangle$ precisely if $n \in X_i \setminus X_{i+1}$.
Let $\langle B_n \rangle_{n \in \mathbb{N}}$ be the sequence of correct responses by Bob, generated by $\mathrm{SeqP}_k(A, B, R)$.

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Proof.

Define $\langle Y_1, \ldots, Y_{k-2} \rangle$ as follows:

$y \in Y_i$ if $\langle a_0, \ldots, a_i \rangle \; R \; \langle b_0, \ldots, b_i \rangle_y$ fails but $\langle a_0, \ldots a_{i'} \rangle \; R \; \langle b_0, \ldots b_{i'} \rangle_y$ holds for all $i' < i$.

$y \in X_{i-1} \setminus (Y_1 \cup \cdots \cup Y_i)$ if $\langle a_0, \ldots, a_{i'} \rangle \; R \; \langle b_0, \ldots, b_{i'} \rangle_y$ holds for all $i' \leq i$.

$\Rightarrow X_{i-1} \setminus (Y_1 \cup \cdots \cup Y_i)$ is enumerable.

$Y_i \subseteq X_i$ since $\bar{b}_y$ is a winning response.

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Proof.

By the hypothesis that $\langle X_0, \ldots X_{k-1} \rangle$ is a good k-tuple, we know that there exists an element $y \in X_{k-1} \setminus (Y_1 \cup \cdots \cup Y_i)$.

So in B_y, $\langle a_0, \ldots, a_{k-2} \rangle \; R \; \langle b_0, \ldots, b_{k-2} \rangle$ holds and in fact $\langle a_0, \ldots, a_j \rangle \; R \; \langle b_0 \ldots, b_j \rangle$ holds for all $j \leq k-1$, contradicting that (A, B, R') is not k-solvable.

$\square$